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ITERATION OF SOUSLIN FORCING, PROJECTIVE MEASURABILITY AND THE BOREL CONJECTURE

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ABSTRACT

From an inaccessible cardinal we construct a model of ZFC where the Borel Conjecture holds and all projective sets of reals are measurable. This continues the investigation of countable support iterations of Proper Souslin forcing notions, started in a paper of Judah and Shelah.

Introduction

It is well known that all analytic (Σ^1_1) sets of reals are Lebesgue measurable, but the axioms of ZFC do not decide whether more complex projective sets are Lebesgue measurable: In Gödel's constructible universe L there are already nonmeasurable Δ^1_2 sets. On the other hand, in the celebrated model of Solovay

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[15], all projective sets of reals are Lebesgue measurable, and satisfy many other regularity properties (have the Baire property, the Ramsey property, etc.).

Writing PM for "all projective sets of reals are Lebesgue measurable", PB for "all projective sets of reals have the property of Baire", $\Sigma^1_n(M)$ for "all Σ^1_n-sets of reals are Lebesgue measurable", etc., and IC for "there exists an inaccessible cardinal", Solovay showed:

If Con(ZFC+IC), then Con(ZFC+PM).

Conversely, Shelah showed in [14]:

If Con(ZFC+Σ^1_3(M)), then Con(ZFC+IC).

A set X of reals is said to be "of strong measure zero", if for every sequence $\langle \varepsilon_k : k \in \omega \rangle$ of positive numbers there exists a sequence of reals $\langle x_k : k \in \omega \rangle$ with $X \subseteq \bigcup_{k \in \omega}(x_k - \varepsilon_k, x_k + \varepsilon_k)$. It is clear that every countable set is of strong measure zero, and every strong measure zero set has Lebesgue measure zero. While this second implication cannot be reversed (the Cantor set or indeed any set containing a perfect set is not of strong measure zero), the question whether the first implication can be improved to an equivalence turned out to be undecidable in ZFC. Let BC (Borel's Conjecture) stand for the statement

Every strong measure zero set is countable.

Then CH (the continuum hypothesis) implies ¬BC, but Laver [10] constructed a model in which BC holds.

In [6], Judah showed that under BC, Shelah's Σ^1_3(M) can be replaced by Δ^1_2(B):

If Con(ZFC+BC+Δ^1_2(B)), then Con(ZFC+IC).

(The same is not true for Δ^1_2(M), because Laver's construction, followed by $\aleph_1$ many random reals, will yield a model satisfying BC (by [9]) and Δ^1_2(M) (by [4])).

For the other direction, Judah and Shelah showed in [5]:

If Con(ZFC+IC), then Con(ZFC+BC+Σ^1_2(M)).

In section 4 we will get the following result as a corollary to the Main Theorem:

0.1 Theorem:

If Con(ZFC+IC), *then* Con(ZFC+PM+BC)

so that the following equiconsistencies hold:

(1) Con(ZFC+PM+BC) *iff* Con(ZFC+PM+¬BC) *iff* Con(ZFC+IC).

(2) Con(ZFC+¬PM+BC) *iff* Con(ZFC+¬PM+¬BC) *iff* Con(ZFC).

The contents of sections 1–4 are as follows: In section 1 we discuss Souslin proper forcing notions, and we show how to get a version of Martin's axiom for these forcing notions. Judah and Shelah [5] called a forcing notion Q a "Souslin forcing" if its underlying set is a Σ^1_1 subset of $\mathbb{R}$ (the set of reals) and also $\leq$ and $\perp$ relations are Σ^1_1. Q is called "Souslin proper", if it is Souslin, proper, and moreover, for any countable model M satisfying some large fragment of ZFC (not necessarily an elementary submodel of some $H(\chi)$) and for every $p \in Q^M$, there exist a condition $q \geq p$ in Q forcing that $G \cap M$ is Q^M-generic over M (where $Q^M = Q \cap M$ is a Souslin forcing in M that uses the definitions of Q).

Unfortunately, for many tree-like forcing notions the $\perp$-relation is not analytic. (We wish to thank Uri Abraham for pointing out this problem.) If we want to include also these forcing notions, we have to amend the definition of "Souslin forcing" to admit forcing notions in which only $\leq$ is analytic, but which satisfy the strengthened properness condition (mentioned above) not only in V but also in every model of ZFC.

In section 2 we deal with countable support iteration of Souslin proper forcing notions. We generalize the operation that sends a generic $G \subseteq Q$ to $G \cap M \subseteq Q^M$ to apply to filters $G_\alpha \subseteq P_\alpha$ on iterations of Souslin proper forcing notions, and we prove (essentially) that for every $p \in P^M_\alpha$ (the iteration, computed in M) there exists a $q \in P_\alpha$ forcing that the "restriction" of G to M is generic and contains p. (For the technicalities, in particular what "restriction" and "essentially" mean, see section 2.) To ease the understanding of this proof, we first give a structurally similar (but technically simpler) proof of "properness is preserved under countable support iteration."

The results of this section are similar to those in [5]. We give definitions that we think are more natural than the original definitions. (We rely less on definition by simultaneous induction, and more on the existential completeness lemma.) The proof of theorem 4.5 in [5] is incomplete, but a stronger version of the theorem it was supposed to prove is given in Theorem 0.1 of this paper.

In section 3, we use the result of section 2 to show

0.2 THEOREM: *Let $R_\kappa = Coll(<\kappa, \aleph_0)$, the Lévy collapse of an inaccessible cardinal κ. Let $\underset{\sim}{\overline{P}} = \langle P_i, Q_i : i < \kappa^+ \rangle$ be an R_κ-name for a countable support iteration of Souslin proper forcing of length $\kappa^+ = \aleph_2^{V^{R_\kappa}}$, $P_{\kappa^+} = \lim P_i$.*

Then $\Vdash_{R_\kappa * P_{\kappa^+}}$ "for every real x, there exists a forcing notion $Q \in V$ of size less than κ, and a V-generic $H \subseteq Q$ such that $x \in V[H]$."

Finally in section 4, we repeatedly apply this theorem to all reals in $V^{R_\kappa * P_{\kappa^+}}$ (after collapsing κ^+ to κ, so there will be only κ many reals), to get

0.3 COROLLARY: *Assume that R_κ, $\underset{\sim}{\overline{P}}$ are as above, and let $\underset{\sim}{C}$ be the forcing notion in $V^{R_\kappa * P_{\kappa^+}}$ collapsing κ^+ $(= \aleph_2)$ to κ $(= \aleph_1)$ using countable$^{V^{R_\kappa * P_{\kappa^+}}}$ conditions. Then if $G \subseteq R_\kappa * P_\kappa * C$ is generic (letting $V_{\kappa^+} = V[G \cap R_\kappa * P_{\kappa^+}]$, $V_{\kappa^++1} = V[G]$) in V_{κ^++1} there exists a V-generic $H'_\kappa \subseteq R_\kappa$ such that*

$$\mathbb{R}^{V[H'_\kappa]} = \mathbb{R}^{V_{\kappa^+}} = \mathbb{R}^{V_{\kappa^++1}}.$$

(This consequence of 0.2 is a special case of a more general theorem of [18], but we prove it anyway in section 3.)

As a consequence, the first order theory of reals in V_{κ^+} is the same as the first order theory of the reals in $V[H_\kappa]$, and in particular $V_{\kappa^+} \models PM$.

If we let our iteration $\underset{\sim}{\overline{P}}$ be an iteration of Mathias (or Laver) forcing, we get $V_{\kappa^+} \models BC$, and hence 0.1.

If φ is a property of forcing notions, we write MA(φ) for the statement

> For all forcing notions Q satisfying φ, for all collections, $\langle D_i : i < \omega_1 \rangle$ of $\aleph_1$ many dense open sets of Q, there exists a filter $G \subseteq Q$ such that, for all $i < \omega_1$, $G \cap D_i \neq \emptyset$.

Harrington and Shelah [2] proved that MA(ccc), together with any of PM, PB, Σ^1_3(M), or Δ^1_3(B), is equiconsistent with the existence of a weakly compact cardinal. Judah and Shelah [8] later lowered Σ^1_3(M) to Δ^1_3(M), and they also showed that MA(Souslin ccc), together with any of PM, PB, Σ^1_3(M), or Σ^1_3(B), is equiconsistent with the existence of an inaccessible cardinal.

In section 1, we give a reflection argument to show how to obtain MA(Souslin proper) by a countable support iteration of length $\aleph_2$. A bookkeeping argument (as in [16]) enables us to let the iterands Q_i range over all Souslin proper forcing notions.

As a consequence, we get in section 4

0.4 COROLLARY: *Assume* Con(ZFC+IC). *Then*

$$(\text{ZFC} + \text{MA(Souslin Proper)} + 2^{\aleph_0} = \aleph_2 + \text{PM})$$

is consistent.

0.5 Historical Remarks: It should be pointed out that ours is not the first model of PM+BC. By a result of Woodin [18], the mere existence of a supercompact cardinal already implies PM. Since a supercompact cardinal cannot be destroyed by a small forcing such as Laver's iteration,

(*) Con(ZFC+$\exists$ supercompact) implies Con(ZFC+PM+BC).

Of course, in the above statement, "BC" can be replaced by any other statement that can be forced with a small forcing notion. Also, by a result of Martin and Steel [11], supercompact cardinals imply full projective determinacy (PD), so after forcing with Laver's iteration we could even get PD+BC.

Similarly, the consistency of large cardinals easily implies the consistency of MA(Souslin proper)+PM, or even MA(proper)+PD.

While (*) gives only a glimpse of the awesome power of large cardinal hypotheses, we believe that to kill a fly, a fly-swatter is more appropriate than a nuclear warhead. The main point in this paper is 0.2 and its corollary 0.3: If we start from Solovay's model of PM, iteration of Souslin proper forcing notions does not change the first order theory of the reals. Thus the consistency strength of PM+BC can be shown to be exactly the same as that of PM, namely, it requires just one inaccessible cardinal.

We conclude the introduction by setting up a notation and quoting a few well known facts:

0.6 Notation: For a forcing P, G_P is (depending on the context) either $= \{\langle p,p\rangle : p \in P\}$, the canonical name for the generic object (also called the generic filter) added by P, or a variable ranging over all V-generic filters $G \subseteq P$.

We interpret $p \leq q$ as "q extends p", and we write $q \geq^* p$ if $q \Vdash_P p \in G_P$.

$\emptyset_P$ is always the weakest condition in the forcing P.

We will consider a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ of forcing notions, i.e. conditions in $r \in P_\alpha$ are countable partial functions with domain a subset of α, and for each $\beta \in \mathrm{dom}(r)$, $\Vdash_{P_\beta} r(\beta) \in Q_\beta$, where, for all α, Q_α is a P_α-name of a forcing notion.

For $\beta \notin \mathrm{dom}(r)$, we let $r(\beta) = \emptyset_{Q_\beta}$.

We may write G_α for G_{P_α}.

When we talk about a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$, it is understood that P_ε is defined as the countable support limit of this iteration.

If $\beta \leq \alpha$, and $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is an iteration, then G_β always denotes $G_\alpha \cap P_\beta$.

When we fix a ground model $V = V_0$, and consider an iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle \in V_0$, we write V_α for $V[G_\alpha]$. **Note that this conflicts with the notation that some authors use for the sets of rank $< \alpha$.**

For a P-name $\underset{\sim}{x}$ and a generic $G \subseteq P$, we let $\underset{\sim}{x}[G]$ be the evaluation of $\underset{\sim}{x}$ by G. But if M is a model of a large fragment of ZFC, then we let

$$M[G] := \{\underset{\sim}{x}[G] : M \models \underset{\sim}{x} \text{ is a } P\text{-name}\}.$$

$\min A \cup B$ should be read as $\min(A \cup B)$. For a successor ordinal μ, $\mu - 1$ denotes its predecessor.

0.7 Notation: We let ZFC* be a sufficiently large fragment of ZFC, excluding the power set axiom $\forall x \exists \mathfrak{P}(x)$, but including the statement

$$\text{The set } \{\omega, \mathfrak{P}(\omega), \mathfrak{P}(\mathfrak{P}(\omega)), \dots\} \text{ exists.}$$

We will consider countable transitive models $(M, \in)$ satisfying ZFC*.

The following four facts are well known:

0.8 Fact and Notation: We consider an iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$. If $G_\alpha \subseteq P_\alpha$ is generic, and $H \subseteq Q_\alpha[G_\alpha]$ is any set, we let

$$G_\alpha * H = \{r \in P_{\alpha+1} : r{\restriction}\alpha \in G_\alpha,\ r(\alpha)[G_\alpha] \in H\}.$$

Then: $G_\alpha * H$ is generic iff G_α is generic and $H \subseteq Q_\alpha[G_\alpha]$ is generic over $V[G_\alpha]$.

Conversely, writing $G(\alpha)$ for $\{q(\alpha)[G_\alpha] : q \in G_{\alpha+1}\}$, we know that it is a generic filter on $Q_\alpha[G_\alpha]$ over the model $V[G_\alpha]$, and $G_{\alpha+1} = G_\alpha * G(\alpha)$.

0.9 Fact: If $\beta > \alpha$, $q \in P_\alpha$, $p \in P_\beta$, $q \geq^* p{\restriction}\alpha$, then $q^+ := q \cup p{\restriction}[\alpha, \beta)$ is in P_β, and $q^+ \geq^* p$.

0.10 "Existential Completeness Lemma": *For any forcing P, and any condition $p \in P$, any formula $\varphi(x)$:*

$$p \Vdash \exists x\, \varphi(x) \quad \text{iff} \quad \text{there is a name } \underset{\sim}{\tau} \text{ such that } p \Vdash \varphi(\underset{\sim}{\tau}).$$

0.11 Fact: Assume that λ is a limit ordinal. Then for a generic $G_\lambda \subseteq P_\lambda$, for all $p \in P_\lambda$,

$$p \in G_\lambda \Leftrightarrow \forall \alpha < \lambda\ p{\restriction}\alpha \in G_\alpha.$$

0.12 Note: If the reader is only interested in Theorem 0.1, and not in 0.4 and the general theory of Souslin proper forcing, he/she can use the following dictionary to simplify the paper:

d codes Q	$Q =$ Laver forcing
Q_d, Q_{d_α}, Q_α	Laver forcing (in V^{P_α})
P_α, $P_{\vec{d}\restriction\alpha}$, etc.	a countable support iteration of Laver forcing of length α
d_α, c_α, e_α	0 (actually, a P_α-name for 0)
$\vec{d}$, $\vec{d}\restriction\alpha$, $\vec{c}$, etc.	a sequence of ε (or α, etc.) many zeroes

After having replaced all phrases by their counterparts in the above dictionary, the reader should not be surprised if some lemmata or some cases within proofs become vacuously true. For example, "$d_\alpha[G_\alpha] = c_{\alpha s}[G'_\alpha]$ codes a Souslin proper forcing notion" (in 2.8) reduces to "0=0, and Laver forcing is Souslin proper", which is just true. Also, $\vec{d}$ and $\vec{c}$ will automatically be "corresponding" sequences (see 2.12).

However, the proofs in the "interesting" cases (see 2.11 and 2.18) stay of essentially the same complexity.

1. Souslin Proper Forcing

1.1 Notation: Using a universal Σ^1_1-set, we can associate with each real d two Σ^1_1 relations $\leq_d$ and $\perp_d$ (subsets of $\mathbb{R} \times \mathbb{R}$) such that every analytic pair $\leq, \perp$ appears as some $\leq_d$, $\perp_d$, and the relations $x \leq_d y$ and $x \perp_d y$ are Σ^1_1.

1.2 Definition: We say that "d codes a strongly Souslin forcing" iff

(1) $Q_d := \langle \text{field}(\leq_d), \leq_d \rangle$ is a partial quasiorder. (We also write Q_d for the underlying set field$(\leq_d)$.)

(2) For all $x, y \in Q_d$: $x \perp_d y \Leftrightarrow \neg\exists z \in Q_d : x \leq_d z \,\&\, y \leq_d z$.

1.3 Definition: We say that "d codes a Souslin forcing" iff

$$Q_d := \langle \text{field}(\leq_d), \leq_d \rangle \text{ is a partial quasiorder.}$$

(Here we disregard the relation $\perp_d$.)

1.4 Remark: Clearly these are Π^1_2 conditions on d.

1.5 Definition: Assume that d codes a Souslin forcing Q_d, and M is a model of ZFC* that contains d.

(1) We let Q^M_d be the Souslin forcing coded by d in M.

(2) If $p \in Q_d^M$, $q \in Q_d$, we say that q is (p, M)-generic, iff

$$q \Vdash_{Q_d} \text{“}G_{Q_d} \cap M \text{ is } Q_d^M\text{-generic over } M \text{ and contains } p.\text{”}$$

(3) We say that “Q_d is a strongly Souslin proper forcing” or “d codes a strongly Souslin proper forcing” if d codes a strongly Souslin forcing, and

(∗) for all countable M as above, every $p \in Q_d^M$ there exists a (p, M)-generic $q \in Q_d$

(4) We say that “Q_d is a Souslin proper forcing” or “d codes a Souslin proper forcing” if:
   (a) Either d codes a strongly Souslin proper forcing, (so in particular, $\perp_d = \perp_{(Q_d, \leq_d)}$),
   (b) or d codes a Souslin forcing and there is a proof in ZFC^* that (∗) holds.

1.6 Remark: Since we consider arbitrary models of ZFC^* rather than only elementary submodels of some large initial segment of V, the statement “Q is a strongly Souslin proper forcing” is (apparently) stronger than “Q is a Souslin forcing and Q is proper”. (We do not know if these two notions are in fact equivalent.)

1.7 Remark: (1) “d codes a Souslin proper forcing” is a Π^1_3 statement about d. Hence (by Shoenfield's absoluteness theorem) if it holds in V, it holds in every submodel that contains all countable ordinals.

(2) If $(M, \in)$ is a transitive model of a sufficiently large part of ZFC (M may be a class), and $M \models$“$\chi := |\bigcup\{\omega, \mathfrak{P}(\omega), \mathfrak{P}(\mathfrak{P}(\omega)), \dots\}|$ and χ^+ exist,” and $M_0 := H(\chi^+)^M$ is countable, then M_0 is a countable model of ZFC*, and q is (p, M)-generic iff q is (p, M_0)-generic. So for all practical purposes we can pretend that M is countable. (In particular this is true if ω_1 is a strongly inaccessible cardinal in M.)

Proof of (1): Every countable model M is isomorphic to some well-founded (ω, R). If $x \in \mathbb{R}^M$, we also write x for its image under this isomorphism.

It is enough to show that “d codes a strongly Souslin proper forcing notion” is a Π^1_3 statement.

d codes a strongly Souslin proper forcing iff <u>for all</u> $R \subseteq \omega \times \omega$

Either (ω, R) is not well-founded (i.e., <u>there exists</u> an R-descending sequence),

or $(\omega, R) \not\models ZFC^*$ (this is Δ^1_1),

or $d \notin (\omega, R)$

or for all $p \in Q_d^M$ <u>there is</u> $q \in Q_d$ such that <u>for all</u> $r \geq q$

(i) for all D such that $(\omega, R) \models$ "D is open dense in Q_d", there is an i, $(\omega, R) \models i \in D$, $r \not\perp i$,

(ii) $r \not\perp p$.

((i) implies that $q \Vdash G \cap D \neq \emptyset$, and (ii) implies that $q \Vdash p \in G$.) ∎

Proof of (2): M and M_0 contain the same dense sets of Q_d. ∎

1.8 Example: Mathias forcing is (strongly) Souslin proper.

Proof: see [5]. ∎

1.9 Example: Laver forcing is Souslin proper.

We will prove this after collecting a few facts and definitions about Laver forcing.

1.10 Definition: Recall that for $p \subseteq \omega^{<\omega}$, $\eta \in p$, we let

$$\mathrm{Succ}_p(\eta) = \{\nu \in p : \eta \subset \nu, |\nu| = |\eta| + 1\}$$

and

$$p^{[\eta]} = \{\nu \in p : \nu \subseteq \eta \vee \eta \subseteq \nu\}.$$

The following definition is from [10].

1.11 Definition: Laver forcing $\mathbb{L}$ is the set of all trees $p \subseteq \omega^{<\omega}$, such that for some $\eta_0 \in p$, $p = p^{[\eta_0]}$ and for all $\nu \in p$ with $\nu \supseteq \eta_0$, $\mathrm{Succ}_p(\nu)$ is infinite. We call this η_0 the stem of p, $\eta_0 = \mathrm{stem}(p)$).

$p \leq q$ (or q extends p) iff $p \supseteq q$.

$p \leq_n q$ (or q n-extends p) iff $p \leq q$ and $p \cap \omega^{\leq n} = q \cap \omega^{\leq n}$.

For $p \leq q$ we say that q is a pure extension of p iff $\mathrm{stem}(p) = \mathrm{stem}(q)$.

We let $\mathbb{L}_n = \{p \in \mathbb{L} : |\mathrm{stem}(p)| \geq n\}$. This is a dense open subset of $\mathbb{L}$.

1.12 Definition: For $p \in \mathbb{L}$,

(1) $b \subseteq p$ is a branch, if b is a maximal linearly $\subseteq$-ordered set (iff for some $f : \omega \to \omega$, $b = \{f{\restriction}n : n \in \omega\}$).

(2) $A \subseteq p$ is an antichain iff for all $\eta \neq \nu$ in A, $\eta \not\subseteq \nu$ and $\nu \not\subseteq \eta$.

(3) $F \subseteq p$ is a front, if F is an antichain and every branch of p meets F.
(4) $F \subseteq p$ is a front above n iff F is a front and $\forall \eta \in F\ |\eta| \geq n$.

1.13 Fact: Assume that $F \subseteq p \in \mathbb{L}$ is a front above n, $n \geq |\mathrm{stem}(p)|$, and for all $\eta \in F$, q_η is a pure extension of $p^{[\eta]}$ (so $\mathrm{stem}(q_\eta) = \eta$).

Then $q := \bigcup_{\eta \in F} q_\eta$ is a condition, $p \leq_n q$ and $q^{[\eta]} = q_\eta$.

1.14 Notation: For $D \subseteq \mathbb{L}$, $p \in \mathbb{L}$, we say $p \in^* D$ if there is a pure extension of p that is in D.

1.15 LEMMA: *Assume $p \in L$, $|\mathrm{stem}(p)| < n$, D is dense open, $D \subseteq \mathbb{L}_n$. Then there exists a condition $q = q(p, n, D) \geq_n p$ and a front $F = F(p, n, D) \subseteq q$ such that for all $\eta \in F$, $q^{[\eta]} \in D$. (Since $q \cap \omega^{\leq n} = p \cap \omega^{\leq n}$ is infinite, we must have $\mathrm{stem}(p) = \mathrm{stem}(q)$.)*

Proof: Consider the following game $G(n, p, D)$: the game lasts ω many moves. In the k-th move, player I plays $\eta_k \in {}^k\omega$, and player II answers with a set $A_k \subseteq {}^{k+1}\omega$. They must obey the following rules:

(1) For $k \geq 1$, player I must play $\eta_k \in A_{k-1}$. (Since $\eta_0 \in {}^0\omega$, we must have $\eta_0 = \emptyset$.)
(2) For $k < n$, player II must play $A_k = \mathrm{Succ}_p(\eta_k)$.
(3) For $k \geq n$, player II must play an infinite set $A_k \subseteq \mathrm{Succ}_p(\eta_k)$. (We can even require $\mathrm{Succ}_p(\eta_k) - A_k$ finite.)

Player II wins if for some k, $p^{[\eta_k]} \in^* D$.

We claim that player I cannot have a winning strategy. For assume that σ is a winning strategy for I. Let

$$\bar{p} := \{\eta : \eta \text{ appears in a play in which } \sigma \text{ was used}\}.$$

Then $\bar{p}$ is a condition (this follows easily from (3)), $p \leq_n \bar{p}$, and for all $\eta \in \bar{p}$, $p^{[\eta]} \notin^* D$. This contradicts the fact that D is dense.

Since the game is closed, it is determined, so player II must have a winning strategy σ. Again, let

$$\bar{p} := \{\eta : \eta \text{ appears in a play in which } \sigma \text{ was used}\}$$

and let

$$F := \{\eta \in \bar{p} : \bar{p}^{[\eta]} \in^* D \ \ \& \ \ \forall \nu \subset \eta\ \bar{p}^{[\nu]} \notin^* D\}.$$

Then F is an antichain, and since every branch through $\bar{p}$ corresponds to a play in which II wins, F is a front. Since $D \subseteq \mathbb{L}_n$, F is above n.

For each $\eta \in F$ let $q_\eta \in D$ be a pure extension of $\bar{p}^{[\eta]}$, and let $q = \bigcup_{\eta \in F} q_\eta$ as in 1.13. Then $p \leq_n q$, and for all $\eta \in F$, $q^{[\eta]} = q_\eta \in D$. ∎

1.16 Proof of 1.9: It is clear that Laver forcing is Souslin (though not necessarily strongly Souslin). We will give a proof (in ZFC*) that Laver forcing always satisfies the properness condition $(*)$ in 1.5(3).

Let M be a countable model. Let $\langle D_n : n \in \omega \rangle$ be a sequence of sets $D_n \subseteq \mathbb{L}_n$, such that for all n, $M \models$"D_n is open dense", and for every set $D \in M$ such that $M \models$"D is open dense", there is an n such that $M \models D_n = D \cap \mathbb{L}_n$.

Fix a condition $p \in \mathbb{L} \cap M$, $|\text{stem}(p)| + 1 = k_0$. We will define two sequences $\langle p_n : k_0 \leq n < \omega \rangle$ and $\langle F_n : k_0 + 1 \leq n < \omega \rangle$ such that $p_{k_0} = p$, for all n, $p_n \leq_n p_{n+1} \in M$, F_{n+1} is a front above n in p_{n+1}, and for all $\eta \in F_{n+1}$, $p_{n+1}^{[\eta]} \in D_n$.

Given p_n, we let p_{n+1} and F_{n+1} be obtained from p_n as in 1.15.

Clearly $q := \bigcap_n p_n$ is a condition in $\mathbb{L}$, extending all conditions p_n.

Let $G \subseteq \mathbb{L}$ be generic, $q \in G$, then we claim $p \in G$ (this is clear) and G is generic over M, i.e., G meets every D that is a dense subset of $\mathbb{L}$ in M. For this it is enough to meet every D_n.

Fix n. G defines a branch g through q:

$$\eta \in g \Leftrightarrow q^{[\eta]} \in G.$$

g is also a branch through p_{n+1}, so by 1.15, g meets F_{n+1}. (Being a front is a Π^1_1-property, hence F_{n+1} is still a front in $V[G]$.) Let $\eta \in F_{n+1} \cap g$. Then $p_{n+1}^{[\eta]} \in D_n \cap G$.

This finishes the proof that Laver forcing is Souslin proper. ∎

We conclude this section by showing how to obtain the consistency of Martin's Axiom for Souslin proper forcing notions with the negation of CH. In our model the size of the continuum will be $\aleph_2$.

In this context, we should mention the following question:

1.17 Problem: Is MA(Souslin proper$+2^{\aleph_0} > \aleph_2$) consistent?

1.18 THEOREM: *Assume* CH *and* $2^{\aleph_1} = \aleph_2$. *Then there is a countable support iteration* $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ *of length* $\aleph_2$ *of Souslin proper forcing notions* Q_α *such that*

$$P_{\omega_2} \Vdash \text{“MA(Souslin proper)} + 2^{\aleph_0} = \aleph_2\text{”}.$$

1.19 Definition:

$S_2^1 := \{\alpha < \omega_2 : cf(\alpha) = \omega_1\}$.

$S_2^0 := \{\alpha < \omega_2 : cf(\alpha) = \omega_0\}$.

S_2^0 and S_2^1 are stationary.

We will use the following well known result of Solovay [17]:

1.20 LEMMA: *There exists a sequence of $\aleph_2$ many disjoint stationary subsets of S_2^1.*

1.21 Notation: We will use a sequence $\langle S_\gamma : \gamma < \omega_2 \rangle$ as in 1.20. Also, we fix a bijection $\Gamma : \omega_2 \times \omega_1 \to \omega_2$. Wlog we may assume that all elements of $S_{\Gamma(\beta,i)}$ are $\geq \beta$.

1.22 Fact: (Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$.) If $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon \rangle$ is a countable support iteration of Souslin proper forcing notions of length $\varepsilon \leq \omega_2$, then

(1) P_ε satisfies the $\aleph_2$-cc and $P_\varepsilon \Vdash$ CH.

(2) If $S \subseteq \aleph_2$ is stationary, then $\Vdash_\varepsilon$"S is stationary".

Proof: (1) is well known from [13] (and true even if the Q_α are just proper and of size $\leq \aleph_1$). (2) is a well known consequence of (1). ∎

1.23 Definition: We define a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ of Souslin proper forcing notions Q_α as follows:

In V^{P_α}, we enumerate $\mathbb{R} = \{r_i^\alpha : i < \omega_1\}$.

Assume that $\alpha \in S_\gamma$, $\gamma = \Gamma(\beta, i)$ (so $\beta \leq \alpha$), and r_i^β codes a Souslin proper forcing, then $Q_\alpha = Q_{r_i^\beta}$.

If $\alpha \notin S_\gamma$ for all γ, or r_i^β does not define a Souslin proper forcing, then let Q_α be some fixed Souslin proper forcing notion adding a real, say Cohen forcing. (We could even let $Q_\alpha = \{\emptyset\}$ in this case.)

(I.e. we let Q_α be a P_α-name such that the previous two paragraphs are forced about Q_α.)

Clearly $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ is a countable support iteration of Souslin proper forcing notions.

We claim that $\Vdash_{\omega_2}$"MA(Souslin proper)". Since we add reals in cofinally many stages, $\Vdash 2^{\aleph_0} = \aleph_2$.

1.24 LEMMA: *Assume $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ is the iteration from 1.23, $G_{\omega_2} \subseteq P_{\omega_2}$ is generic, d codes a Souslin proper forcing Q_d in $V[G_{\omega_2}]$, A a maximal antichain* *in Q, say $A = \{a_i : i < \omega_2\}$, where $a_i = \underset{\sim}{a}_i[G_{\omega_2}]$. Wlog each $\underset{\sim}{a}_i$ is a P_α-name for some $\alpha < \omega_2$. For $\alpha \leq \omega_2$, let $V_\alpha = V[G_\alpha]$, $A^\alpha = \{a_i : i < \alpha\}$.*

Then in V_{ω_2}, there exists a closed unbounded set $C_A \subseteq \omega_2$ such that for all $\alpha \in C_A \cap S^1_2$, $A^\alpha \in V_\alpha$ and

$$V_\alpha \models A^\alpha \text{ is a maximal antichain of } Q_d.$$

Proof: For some α we have $d \in V_\alpha$. For notational simplicity only, assume that d is in the ground model V_0.

In V_{ω_2}, define a function $f : \omega_2 \to \omega_2$ by $f(\alpha) =$ some β such that

(1) $\forall i < \alpha$: a_i is a P_β-name.

(2) If there is $a \in V_\alpha$ such that $A^\alpha \cup \{a\}$ is an antichain, then there is $j < \beta$ such that a is compatible with a_j.

Clearly it is possible to define such a function f. Note that the compatibility relation is analytic and hence absolute.

Now let $C_A := \{\delta < \omega_2 : \forall\alpha < \delta : f(\alpha) < \delta\}$. Clearly C_A is closed unbounded.

We claim that for all $\delta \in C_A$, A^δ is in V_δ. To prove this, note that $A^\delta = \{a_i : i < \delta\} = \{\underset{\sim}{a}_i[G_\delta] : i < \delta\}$, and $\{\underset{\sim}{a}_i : i < \delta\} \in V_0$.

It remains to prove that for $\delta \in C_A \cap S^1_2$,

$$V_\delta \models A^\delta \text{ is a maximal antichain of } Q_d.$$

Assume that this is not true for some δ, so there exists an $a \in V_\delta$ which is incompatible with all a_i, $i < \delta$. Since $cf(\delta) > \omega$, $a \in V_\alpha$ for some $\alpha < \delta$. By definition of the function f there exists a $j < f(\alpha)$ such that a is compatible with a_j. But $j < f(\alpha) < \delta$, a contradiction to the choice of a. ∎

1.25 Proof of 1.18: Consider a Souslin proper forcing Q_d in V_{ω_2}, $d \in V[G_\beta]$, and a set $\{A_k : k < \omega_1\}$ of $\aleph_1$ many maximal antichains of Q_d. Let C_{A_k} be defined as in 1.24, and let $C := \bigcap_{k<\omega_1} C_{A_k}$. Then C is a closed unbounded set.

d, the real that codes the Souslin forcing, appears in some intermediate stage V_β, say $d = r_i^\beta[G_\beta]$. Let $\gamma = \Gamma(\beta, i)$. Then by 1.22, S_γ is still stationary in V_{ω_2}, so there exists $\alpha \in S_\gamma \cap C$. By 1.7, $V_\alpha \models$"d codes a Souslin proper forcing." So by definition of the iteration, $Q_\alpha[G_\alpha] = Q_d^{V_\alpha}$. Since $\alpha \in S^1_2$, by 1.24, A_k^α is a maximal antichain in $Q_d^{V_\alpha}$, so each A_k is met by $G(\alpha)$, the generic subset of Q_α.
∎

2. Iteration of Souslin Proper Forcing

To prove the main theorem (2.16) of this section, we will need the following lemma:

2.1 PRELIMINARY LEMMA: *Assume* $\alpha_1 \le \alpha_2 \le \varepsilon$, $\underset{\sim}{p}_1$ *is a* P_{α_1}*-name for a condition in* P_ε. *Let* D *be a dense open set of* P_ε.

Then $\emptyset \Vdash_{\alpha_2}$ *"there exists a* $p_2 \in P_\varepsilon$ *satisfying (1)–(3)":*

(1) $p_2 \ge \underset{\sim}{p}_1$.
(2) $p_2 \in D$.
(3) *If* $\underset{\sim}{p}_1 \restriction \alpha_2 \in G_{\alpha_2}$, *then* $p_2 \restriction \alpha_2 \in G_{\alpha_2}$.

(By 0.10 there is an α_2-name $\underset{\sim}{p}_2$ for a condition in P_ε such that (1)–(3) are forced about $\underset{\sim}{p}_2$.)

Remark: The P_{α_1}-name $\underset{\sim}{p}_1$ corresponds naturally to a P_{α_2}-name, which we also call $\underset{\sim}{p}_1$. In other words, we may wlog assume that $\alpha_1 = \alpha_2$.

Note that this lemma does not mention properness. It is in fact true for any iteration, since it is really only concerned with the composition of the two forcing notions P_{α_2} and P_β / P_{α_2}.

Proof of the lemma: Assume that the conclusion is false. So there exists a condition $r_2 \in P_{\alpha_2}$ such that

$$r_2 \Vdash \text{there is no } p_2 \text{ satisfying (1)–(3).}$$

We may assume that r_2 decides what $\underset{\sim}{p}_1$ is (i.e. $r_2 \Vdash$"$\underset{\sim}{p}_1 = p_1$", for some $p_1 \in V$), and r_2 also decides whether $p_1 \restriction \alpha_2 \in G_{\alpha_2}$.

CASE 1: $r_2 \Vdash p_1 \restriction \alpha_2 \notin G_{\alpha_2}$: But then (3) is true for any p_2, so

$$r_2 \Vdash_{\alpha_2} \text{there is no } p_2 \text{ satisfying (1)–(2),}$$

which is a contradiction since D is dense open.

CASE 2: $r_2 \Vdash p_1 \restriction \alpha_2 \in G_{\alpha_2}$, i.e. $r_2 \ge^* p_1 \restriction \alpha_2$. Let (by 0.9) $r = r_2 \cup p_1 \restriction [\alpha_2, \varepsilon) \ge^* p_1$, and find $p_2 \in D$, $p_2 \ge r$. Then

$$p_2 \restriction \alpha_2 \Vdash p_2 \text{ satisfies (1)–(3),}$$

again a contradiction, because $p_2 \restriction \alpha_2 \ge r_2$.

To show the significance of this lemma, and also to prepare the reader for the proof in 2.16, we first use this lemma to give a simple proof of the key step in Shelah's well-known theorem: "properness is preserved under countable support iteration."

2.2 INDUCTION LEMMA: *Let $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ be a countable support iteration of proper forcing notions, $N < H(\chi)$ a countable elementary submodel of $H(\chi)$ for some large χ, $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle \in N$.*

For all $\beta \in N \cap \varepsilon$, all $\alpha \in N \cap \beta$, all $\underset{\sim}{p} \in N$: **Assume** *$\underset{\sim}{p}$ is a P_α-name for a condition in P_β, and*

(a) $q \in P_\alpha$,

(b) *q is (P_α, N)-generic.*

(c) $q \Vdash_\alpha \underset{\sim}{p}\restriction\alpha \in G_\alpha \cap N$.

Then *there is a condition q^+:*

(a)$^+$ $q^+ \in P_\beta$, $q^+\restriction\alpha = q$,

(b)$^+$ *q^+ is N-generic,*

(c)$^+$ $q^+ \Vdash_\beta \underset{\sim}{p} \in G_\beta \cap N$.

(For $\alpha = 0$ this shows that properness is preserved in limit stages of countable cofinality.)

The proof is by induction on β.

We omit the (easy) successor step, so we only consider the case where $\beta \in N$ is a limit ordinal. Let $\beta' := \sup(\beta \cap N) = \bigcup_n \alpha_n$, $\alpha = \alpha_0 < \alpha_1 < \cdots$, $\alpha_n \in N$. Let $\langle D_n : n \in \omega\rangle$ enumerate all dense open subsets of P_β that are in N.

First we will define a sequence $\langle \underset{\sim}{p}_n : n \in \omega\rangle$, $\underset{\sim}{p}_n \in N$ such that in N the following will hold:

(0) $\underset{\sim}{p}_n$ is a P_{α_n}-name for a condition in P_β,

(1) $\Vdash_{\alpha_{n+1}} \underset{\sim}{p}_{n+1} \geq^* \underset{\sim}{p}_n$,

(2) $\Vdash_{\alpha_{n+1}} \underset{\sim}{p}_{n+1} \in D_n$,

(3) $\Vdash_{\alpha_{n+1}}$ "If $\underset{\sim}{p}_n\restriction\alpha_{n+1} \in G_{\alpha_{n+1}}$ then $\underset{\sim}{p}_{n+1}\restriction\alpha_{n+1} \in G_{\alpha_{n+1}}$".

For each n we thus get a name $\underset{\sim}{p}_n$ that is in N. For each n we can use the "preliminary lemma" 2.1 in N to obtain $\underset{\sim}{p}_{n+1}$.

Now we define a sequence $\langle q_n : n \in \omega\rangle$, $q_n \in P_{\alpha_n}$, and q_n satisfies (a), (b), (c) (if we write q_n for q, p_n for p, α_n for α).

$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α_n, α_{n+1}, and $\underset{\sim}{p}_n\restriction\alpha_{n+1}$. By (c)$^+$ we know

$$q_n^+ \Vdash (\underset{\sim}{p}_n[G_{\alpha_n}])\restriction\alpha_{n+1} \in G_{\alpha_{n+1}} \cap N.$$

Hence by (3) and the genericity of q_{n+1} we have

$$q_{n+1} \Vdash (\underset{\sim}{p}_{n+1}[G_{\alpha_{n+1}}])\restriction\alpha_{n+1} \in G_{\alpha_{n+1}} \cap N.$$

Since $q_{n+1}\restriction\alpha_n = q_n$, $q = \lim q_n$ exists and is $\geq q_n$ for all n.

We have to show that $q \Vdash \underset{\sim}{p} \in G_\beta \cap N$ and that q is generic. Let G_β be a generic filter containing q. We will write p_n for $\underset{\sim}{p}_n[G_{\alpha_n}]$. (Note that $p_n \in N$, because q_n was N-generic and $q_n \in G_{\alpha_n}$.)

Since $q_n \in G_\beta$, we have $p_n\restriction\alpha_n \in G_{\alpha_n} \cap N$ and $N \models p_n \geq^* p_{n-1} \geq^* \cdots \geq^* p_0$. Hence $p\restriction\alpha_n \in G_{\alpha_n} \cap N$ for all n, and so by 0.11, $p \in G_{\beta'} \cap N = G_\beta \cap N$. Similarly, $p_n \in G_\beta$ for all n.

Consider a dense set $D_n \subseteq P_\beta$. Since $q_{n+1} \Vdash \underset{\sim}{p}_{n+1} \in D_n$, we have $p_{n+1} \in G_\beta \cap D_n \cap N$. Hence q is generic.

2.3 Context: In this whole section, ε will be an ordinal $\leq \omega_2$. S will be a countable subset of ω_2 that is closed under immediate successors and predecessors, where the order type of S is in M. α and γ will stand for ordinals $\leq \varepsilon$ in S. M will be a countable transitive model of ZFC* (0.7) or an "essentially countable" transitive model as in 1.7(2).

For $\alpha \in S$, let α^S be the order type of $\alpha \cap S$.

$\vec{d}$ will be a sequence of length ε, and $\vec{c}$ will be a sequence of length ε^S, $\vec{c} \in M$.

2.4 Definition: Given a sequence $\vec{d} = \langle d_\alpha : \alpha < \varepsilon\rangle$ we can define a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$, by letting Q_α be a P_α-name of $Q_{d_\alpha[G_\alpha]}$ (if this is Souslin proper), i.e. P_α forces the following:

If $d_\alpha[G_\alpha]$ is a code for a Souslin proper forcing, then $Q_\alpha = Q_{d_\alpha}$
otherwise $Q_\alpha = \{\emptyset\}$ is the trivial forcing.

2.5 Notation and Remark: We write $\bar{Q}_{\vec{d}}$ for the iteration defined above, and we write $P_{\vec{d}\restriction\alpha}$ for P_α, i.e., $P_{\vec{d}\restriction\alpha}$ is the α-th iteration stage obtained from the definition $\vec{d}$, and $Q_\alpha = Q_{d_\alpha}$ describes the successor extension. If $\vec{d} \in M \models ZFC^*$, then $P^M_{\vec{d}\restriction\alpha}$ is the α-th iteration stage, computed from the definition $\vec{d}$ in the model M.

(So any sequence $\langle d_\alpha : \alpha < \varepsilon\rangle$ defines a Souslin iteration, but in different models these iterations may look different.)

We will consider sequences $\vec{d}$ and $\vec{c}$, where $|\vec{d}| = \varepsilon$ and $|\vec{c}| = \varepsilon^S$, $\vec{c} \in M$.

2.6 Definition: Assume M, $\vec{c}$, $\vec{d}$, S, ε are as in 2.3.

By induction on $\alpha \in S \cap \varepsilon$, we will define a P_α-name $G_\alpha\restriction(S, M, \vec{c}\restriction\alpha^S, \vec{d}\restriction\alpha)$ (which we usually abbreviate to G'_α or $G_\alpha\restriction(S, M)$), by requiring that $\emptyset_{P_\alpha}$ force the following:

$G'_\alpha \subseteq P^M_{\vec{d}\restriction\alpha^S}$ and

- If $\alpha = \beta + 1$, and
  (a) G'_β is $(P^M_{\beta^S}, M)$-generic,
  (b) $d_\beta[G_\beta] = c_{\beta^S}[G'_\beta]$,
  (c) $d_\beta[G_\beta]$ codes a Souslin Proper forcing (in $V[G_\beta]$);

  then $G'_\alpha = G'_\beta * \Big(G(\beta) \cap M[G'_\beta]\Big)$

  If $\alpha = \beta + 1$ and (a)–(c) does not hold, then $G'_\alpha = \emptyset$.
- If α is a limit, then for $p \in P^M_{\vec{c}\restriction\alpha^S}$ we let

$$p \in G'_\alpha \Leftrightarrow \forall \beta \in S \cap \alpha \; p \restriction \beta^S \in G'_\beta.$$

We also let

$$\mu_\alpha = \mu_\alpha(G_\alpha, S, M, \vec{c}, \vec{d}) := \min(\{\beta \leq \alpha : G'_\beta \text{ not generic}\} \cup \{\alpha + 1\}).$$

Note that the computation of μ_α is absolute for any universe $\supseteq V[G_\alpha]$.

2.7 Remarks and Notation:

(1) Whenever the parameters M, S, $\vec{c}$, $\vec{d}$ are clear from the context we will write G'_α for $G_\alpha \restriction (S, M, \vec{c}\restriction\alpha^S, \vec{d}\restriction\alpha)$, similarly for μ_α.

(2) If G_α is a $P_{\vec{d}\restriction\alpha}$-generic filter, then we also write G'_α for the evaluation of the name G'_α by G_α, similarly for μ_α.

(3) Let $M_\alpha = M[G'_\alpha]$.

(4) It might seem more natural to write $G_\alpha \restriction (S, M)$ as G'_{α^S} $(= (G')_{\alpha^S})$ instead of $G'_\alpha = (G_\alpha)'$, but this would only complicate the notation.

(5) "G'_α is generic" means of course "generic for the forcing $P^M_{\vec{c}\restriction\alpha^S}$ over the model M".

2.8 Fact: If $\alpha \leq \varepsilon$, $G'_\alpha \neq \emptyset$, then for all $\beta < \alpha$, G'_β is generic and $d_\beta[G_\beta] = c_{\beta^S}[G'_\beta]$ codes a Souslin proper forcing. (Whenever we write $d_\beta[G_\beta] = c_{\beta^S}[G'_\beta]$ codes a Souslin proper forcing, this is to be interpreted in $V[G_\beta]$.)

Proof: By definition. ∎

2.9 Fact: Let (in $V[G_\varepsilon]$)

$$\mu = \mu_\varepsilon = \min(\{\alpha \leq \varepsilon : G'_\alpha \text{ not generic}\} \cup \{\varepsilon + 1\}).$$

Then if $\nu + 1 < \mu$,

(a) G'_ν is generic.

(b) $d_\nu[G_\nu] = c_{\nu}s[G'_\nu]$ codes a Souslin proper forcing.

Proof: (a) is by the definition of μ. For (b), note that if $d_\nu[G_\nu] \neq c_{\nu}s[G'_\nu]$, or $d_\nu[G_\nu]$ does not code a Souslin proper forcing, then $G'_{\nu+1} = \emptyset$, so it cannot be generic, contradicting the definition of μ.

2.10 Definition: Assume M, $\vec{c}$, $\vec{d}$, S, ε are as in 2.3, $\gamma \in S$. Assume that $M \models$"$\underset{\sim}{p}$ is a $P^M_{\vec{c}|\gamma s}$-name of a condition in $P^M_{\vec{c}}$". $q \in P_{\vec{d}}$ is called $(\underset{\sim}{p}, \gamma)$-generic, or more exactly $(p, \gamma, M, S, \vec{c}, \vec{d})$-generic iff it forces:

(A) $\mu := \mu_\varepsilon := \min(\{\alpha \leq \varepsilon : \alpha \in S,\ G'_\alpha \text{ is not generic}\} \cup \{\varepsilon + 1\})$ is a successor ordinal.

(B) If $\mu \leq \varepsilon$, then $d_{\mu-1}[G_{\mu-1}] \neq c_{\mu-1}s[G'_{\mu-1}]$, or $V[G_{\mu-1}] \models d_{\mu-1}[G_{\mu-1}]$ does not code a Souslin proper forcing. (For $\nu < \mu - 1$, $d_\nu[G_\nu] = c_{\nu}s[G'_\nu]$ always codes a Souslin proper forcing, by 2.9(b).)

(C) If $\mu > \gamma$, then $\underset{\sim}{p}[G'_\gamma]\restriction(\mu - 1) \in G'_{\mu-1}$.

2.11 Remark and Notation: Usually M, S, $\vec{c}$, $\vec{d}$ are clear from the context, so we can abbreviate $(\underset{\sim}{p}, \gamma, M, S, \vec{c}, \vec{d})$-generic to (p, γ)-generic. (We do not abbreviate it to $\underset{\sim}{p}$-generic, because the value of γ may be important, see (C).)

In applications we usually try to get $\mu_\varepsilon = \varepsilon + 1$. In the proofs below this will be called the "interesting" case, since proofs where $\mu_\varepsilon \leq \varepsilon$ are often trivial.

2.12 Definition: We call $\vec{d}$ and $\vec{c}$ "corresponding" sequences if for all $\alpha < \varepsilon$,

$$\Vdash_\alpha \text{ If } G'_\alpha \text{ is generic over } M, \text{ then } c_{\alpha}s[G'_\alpha] = d_\alpha[G_\alpha]$$

and

$$\Vdash_\alpha d_\alpha[G_\alpha] \text{ codes a proper Souslin forcing.}$$

2.13 Remark: If $\vec{c}$ and $\vec{d}$ are corresponding sequences, then q is $(\underset{\sim}{p}, \gamma)$-generic if it forces:

(AB)' G'_ε is generic over M.

(C)' $\underset{\sim}{p}[G'_\gamma] \in G'_\varepsilon$.

The assumption that $\vec{c}$ and $\vec{d}$ are corresponding sequences would formally simplify the proof of 2.16, but it is not clear if this is sufficient for 3.1. However, those interested only in iteration of Laver forcing (0.12) may take 2.13 as definition of "$(\underset{\sim}{p}, \gamma)$-generic".

2.14 Fact: Assume that G_ε is generic, $\alpha < \varepsilon$, $\mu = \mu_\varepsilon \leq \alpha + 1$, and G_α contains a $(\underset{\sim}{p}\restriction\alpha, \gamma)$-generic condition. $(\alpha, \gamma, \varepsilon \in S.)$

Then 2.10(A) and (C) hold in $V[G_\varepsilon]$, and if moreover $\mu_\varepsilon \leq \alpha$, then also (B) holds.

Proof: In $V[G_\alpha]$, we can compute $\mu_\alpha = \min(\{\beta \leq \alpha : G'_\beta \text{ is not generic}\} \cup \{\alpha + 1\})$, and it will evaluate to μ. Since (A) and (C) depend only on $G_{\mu-1}$, they hold in $V[G_\varepsilon]$ iff they hold in $V[G_{\mu-1}]$ iff they hold in $V[G_\alpha]$. Similarly, if $\mu_\varepsilon \leq \alpha$, then $(\mathrm{B})^{V[G_\varepsilon]}$ is equivalent to $(\mathrm{B})^{V[G_\alpha]}$. ∎

2.15 Fact: If q is $(p, \alpha, \ldots)$-generic, where in M p is a $P_{\vec{c}\restriction\alpha^S}$-name of a condition in $P_{\vec{c}}$, then $q\restriction\alpha$ is $(p\restriction\alpha, \alpha, \ldots)$-generic.

2.16 THEOREM: *Let $\vec{d}$, $\vec{c}$, S, M, γ, ε be as in 2.3. Assume that $M \models$ "$\underset{\sim}{p}$ is a $P_{\vec{c}\restriction\gamma^S}$-name for a condition in $P_{\vec{c}}$" (so $\underset{\sim}{p}\restriction\gamma$ is the name for its restriction to γ, and there is a canonical $P_{\vec{c}}$-name which we also call $\underset{\sim}{p}$).*

Assume that $q \in P_{\vec{d}\restriction\gamma}$ is $(\underset{\sim}{p}\restriction\gamma, \gamma, M, S, \vec{c}\restriction\gamma^S, \vec{d}\restriction\gamma)$-generic. Then there exists a condition $q^+ \in P_{\vec{d}}$ such that $q^+\restriction\gamma = q$ and q^+ is $(\underset{\sim}{p}, \gamma, M, S, \vec{c}, \vec{d})$-generic.

When we apply this theorem, we will only use it for closely related definitions $\vec{c}$ and $\vec{d}$.

2.17 COROLLARY: *Assume $M \models p \in P_{\vec{c}}$. Then there exists a $(p, 0)$-generic condition $q \in P_{\vec{d}}$.*

2.18 Proof of 2.16: The proof is by induction on ε.

SUCCESSOR STEP: Here is the only place where we explicitly use Souslin properness: let $\varepsilon = \alpha + 1$.

Using the induction hypothesis on α; we get a $(\underset{\sim}{p}\restriction\alpha, \gamma)$-generic condition $q^+\restriction\alpha \in P_\alpha$. To find $q^+(\alpha)$, we will work in $V[G_\alpha]$, where G_α is an arbitrary generic filter containing $q^+\restriction\alpha$.

First, let us assume

(a) $G'_\alpha \subseteq P_{\vec{c}\restriction\alpha^S}$ is generic over M,

(b) $d := c_{\alpha^S}[G'_\alpha] = d_\alpha[G_\alpha]$,

(c) $V[G_\alpha] \models d$ codes a Souslin proper forcing.

Then $p(\alpha)[G'_\alpha]$ is in the Souslin proper forcing Q_d, so by definition there exists a condition $q^+(\alpha)$ which is $(p(\alpha)[G'_\alpha], M_\alpha)$-generic.

If (a)–(c) does not hold, then we let $q^+(\alpha) = \emptyset_{Q_\alpha}$.

In any case, we have

(∗) $q^+(\alpha) \in Q_\alpha$, and if (a)–(c), then $q^+(\alpha)$ is $(p(\alpha)[G'_\alpha], M_\alpha)$-generic.

Coming back to V, we use the existential completeness lemma to get a name (which we also call $q^+(\alpha)$) about which (∗) is forced by $q^+ \restriction \alpha$.

Now we have to check that this construction ensures that q^+ is generic.

Assume that $G_{\alpha+1}$ is generic and contains q^+. Let

$$\mu = \mu_{\alpha+1} = \min(\{\beta \le \alpha + 1 : G'_\beta \text{ is not generic}\} \cup \{\alpha + 2\}).$$

If $\mu \le \alpha$, then (since G_α contains $q\restriction\alpha$) the induction hypothesis and 2.14 imply that 2.10(A)–(C) hold in $V[G_{\alpha+1}]$.

If $\mu = \alpha + 1$, then again by 2.14, 2.10(A) and (C) are true. Assume that 2.10(B) does not hold, then $d_\alpha[G_\alpha] = c_{\alpha^S}[G'_\alpha]$ codes a Souslin proper forcing. By (∗), $q^+(\alpha)[G_\alpha]$ is generic, so $G(\alpha) \cap M_\alpha$ is generic over M_α, so by 0.8 we get that $G'_{\alpha+1}$ is generic, contradicting the definition of μ.

If $\mu = \alpha + 2$ (the "interesting case"), then 2.10(A) is true and (B) is vacuously true. Let $p = \underset{\sim}{p}[G'_\gamma]$. Then $p\restriction\alpha \in G'_\alpha$ (because $q^+\restriction\alpha$ was generic) and $p(\alpha) \in G(\alpha)$, so by definition (see 2.6), $p\restriction(\alpha+1) \in G'_{\alpha+1}$. ($d_\alpha[G_\alpha] = c_{\alpha^S}[G'_\alpha]$ codes a Souslin proper forcing, by 2.9(b).)

This concludes the inductive construction for $\varepsilon = \alpha + 1$.

Limit step: Let $\langle \alpha_n : n < \omega \rangle$ be a cofinal sequence in $\varepsilon \cap S$, $\alpha_0 = \gamma$. Let $\langle D_n : n \in \omega \rangle$ enumerate all dense open subsets of $P^M_{\bar{c}}$ that are in M.

First we will define a sequence $\langle \underset{\sim}{p}_n : n \in \omega \rangle$, $\underset{\sim}{p}_n \in M$, $\underset{\sim}{p}_0 = \underset{\sim}{p}$, such that <u>in M</u> the following will hold:

(0) $\underset{\sim}{p}_n$ is a $P_{\bar{c}\restriction\alpha_n^S}$-name for a condition in $P_{\bar{c}}$,

(1) $\Vdash_{\alpha^S_{n+1}} \underset{\sim}{p}_{n+1} \ge \underset{\sim}{p}_n$,

(2) $\Vdash_{\alpha^S_{n+1}} \underset{\sim}{p}_{n+1} \in D_n$,

(3) $\Vdash_{\alpha^S_{n+1}}$ If $\underset{\sim}{p}_n \restriction \alpha^S_{n+1} \in G_{\alpha^S_{n+1}}$ then $\underset{\sim}{p}_{n+1} \restriction \alpha^S_{n+1} \in G_{\alpha^S_{n+1}}$.

(Here, of course, G_β stands for the canonical name (in M) for the generic object of $P^M_{\bar{c}\restriction\beta}$, and $\Vdash_\beta$ is the forcing relation of $P^M_{\bar{c}\restriction\beta}$ in M.)

For each n we thus get a name $\underset{\sim}{p}_n$ that is in M. We use 2.1 and 0.10 (in M) to obtain $\underset{\sim}{p}_{n+1}$.

Now we define a sequence $\langle q_n : n \in \omega \rangle$, $q_n \in P_{\alpha_n}$, $q_0 = q$, such that for all n:

(a) $q_n \in P_{\alpha_n}$, $n \ge 1 \Rightarrow q_n \restriction \alpha_{n-1} = q_{n-1}$.

(b) q_n is $(\underset{\sim}{p}_n \restriction \alpha_n^S, \alpha_n^S)$-generic.

(c) For all $k < n$, q_n is $(\underset{\sim}{p}_k \restriction \alpha_n^S, \alpha_k^S)$-generic.

$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α_n^S, α_{n+1}^S, and $\underset{\sim}{p}_n \restriction \alpha_{n+1}^S$. By induction hypothesis, q_{n+1} is $(\underset{\sim}{p}_n \restriction \alpha_{n+1}^S, \alpha_n^S)$-generic. We have to show:

(b^+) q_{n+1} is $(\underset{\sim}{p}_{n+1} \restriction \alpha_{n+1}^S, \alpha_{n+1}^S)$-generic.

(c^+) For all $k < n+1$, q_{n+1} is $(p_k \restriction \alpha_{n+1}^S, \alpha_k^S)$-generic.

Proof: Let $G_{\alpha_{n+1}}$ be generic containing q_{n+1}, and let

$$\mu = \mu_{\alpha_{n+1}} = \min(\{\beta \le \alpha_{n+1} : G'_\beta \text{ is not generic}\} \cup \{\alpha_{n+1}+1\}).$$

2.10(A)–(B) will always be satisfied, because q_{n+1} is $(\underset{\sim}{p}_n \restriction \alpha_{n+1}^S, \alpha_n^S)$-generic.

Case 1 (the interesting case): $\mu = \alpha_{n+1}+1$. We let $p_k := \underset{\sim}{p}_k[G'_{\alpha_k}]$ for $k \le n+1$. We have to check that $p_k \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}$.

$p_n \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}$, because the $(\underset{\sim}{p}_n \restriction \alpha_{n+1}^S, \alpha_n^S)$-generic condition q_{n+1} is in $G_{\alpha_{n+1}}$.

Since G'_{α_k} is generic over M, the definition of the $\underset{\sim}{p}_k$'s (see 2.1(1)) implies that for $k \le n$,

$$p_k \restriction \alpha_{n+1}^S \le \cdots \le p_n \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}, \text{ so } p_k \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}.$$

For $k = n+1$, by 2.1(3) we have $p_{n+1} \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}$.

Case 2: Assume $\alpha_n < \mu \le \alpha_{n+1}$. Again we let $p_k := \underset{\sim}{p}_k[G'_{\alpha_k}]$ for $k \le n$. The same proof as in case (1) works to show that $p_k \restriction \alpha_{n+1}^S \le \cdots \le p_n \restriction \alpha_{n+1}^S \in G'_{\alpha_{n+1}}$. (By 2.10(C), there is nothing to prove about $\underset{\sim}{p}_{n+1}$.)

Case 3: $\mu \le \alpha_n$. Then also $\mu_{\alpha_n} = \mu$, so we can use the inductive assumptions (c) on q_n, together with 2.14, to show that 2.10(C) is satisfied.

This concludes the construction of the q_n's. Finally, let $q = \lim_n q_n$. We will have finished the proof of 2.16 once we show that q is $(\underset{\sim}{p}, \gamma)$-generic.

Let $q \in G_\varepsilon$.

Case 1: $\mu_\varepsilon = \varepsilon + 1$. Let $p_0 = \underset{\sim}{p}_0[G'_{\alpha_0}]$.

Then for all n, $\mu_{\alpha_n} = \alpha_n + 1$. Since $q_n \in G_{\alpha_n}$, by assumption on the q_n's we know that $p_0 \restriction \alpha_n^S \in G'_{\alpha_n}$, hence $p_0 \in G'_\varepsilon$, by Definition 2.6.

CASE 2: $\mu = \varepsilon$. The whole point of the construction was to ensure that this cannot happen: Since G'_ε is not generic, for some n, $G'_\varepsilon \cap D_n \cap M = \emptyset$. But by construction of $\underset{\sim}{p}_{n+1}$ and since $G'_{\alpha_{n+1}}$ is generic, we know that for all $k \geq n+1$, $p_{n+1} \leq p_k$ (by 2.1(1)). Since $p_k \restriction \alpha_k^S \in G'_{\alpha_k}$, we also have $p_{n+1} \restriction \alpha_k^S \in G'_{\alpha_k}$, and hence $p_{n+1} \in G'_\varepsilon$. Hence $p_{n+1} \in G'_\varepsilon \cap D_n \cap M$.

CASE 3: For some n, $\mu < \alpha_n$. Since G_{α_n} contains the $(\underset{\sim}{p} \restriction \alpha_n)$-generic condition q_n, we are done by 2.14. ∎

3. Application: Iteration of Proper Souslin Forcing in Solovay's Model

In this section we will prove the following

3.1 THEOREM: *Let κ be an inaccessible in V, $2^{2^\kappa} = \kappa^{++}$, $R_\kappa = \mathrm{Coll}(<\kappa, \aleph_0)$ the Lévy collapse. Let $\overline{\underset{\sim}{P}} = \langle P_i, Q_i : i < \kappa^+\rangle$ be a R_κ-name for a countable support iteration of Souslin proper forcing of length $\kappa^+ = \aleph_2^{V^{R_\kappa}}$, $\underset{\sim}{P}_{\kappa^+} = \lim P_i$.*

Then for every V-generic filter $H_\kappa \subseteq R_\kappa$, for every $V[H_\kappa]$-generic filter $G_{\kappa^+} \subseteq \underset{\sim}{P}_{\kappa^+}[H_\kappa]$, for every real $x \in V[H_\kappa * G_{\kappa^+}]$, there exists a forcing notion $Q \in V$ of size less than κ, and a V-generic $H \subseteq Q$ in $V[H_\kappa * G_{\kappa^+}]$ such that $x \in V[H]$.

3.2 Notation: For the proof of 3.1, we will work in $V[H_\kappa]$. We let $P_\varepsilon = \underset{\sim}{P}_\varepsilon[H_\kappa]$.

$R_\kappa = \mathrm{Coll}(<\kappa, \aleph_0)$, and for $\lambda \leq \kappa$ we let $R_\lambda = \mathrm{Coll}(<\lambda, \aleph_0) \lessdot R_\kappa$.

$H_\lambda = H_\kappa \cap R_\lambda$, $V_\lambda = V[H_\lambda]$. Note that for all $\lambda < \kappa$, V_λ is a class in V_κ, but V_λ is an "essentially countable" model as in 1.7(2), so the results of section 2 apply.

Idea of the proof: We will find a sufficiently large $\lambda < \kappa$ and a countable support iteration of Souslin proper forcing notions in V_λ (of length $< \omega_2^{V_\lambda} < \kappa$), which resembles an initial segment of the iteration $\overline{P}$. Then we use an initial segment of the filter G_{κ^+} to obtain a V_λ-generic filter G'. It will turn out that $x \in V[H_\lambda * G']$.

3.3 Proof of 3.1: Assume x is a real in $V[H_\kappa * G_{\kappa^+}]$. Then in $V[H_\kappa]$ there exists an $\varepsilon < \kappa^+$ and a P_ε-name $\underset{\sim}{x}$ and a P_ε-condition p such that $p \Vdash$ "$\underset{\sim}{x}$ is a real", and $\underset{\sim}{x}[G_{\kappa^+}] = x$. Assume, towards a contradiction, that p forces that there exists no such forcing Q as promised in 3.1.

We work in $V[H_\kappa]$. Let $N < H(\chi)$ for some large χ be a countable model containing p, $\underset{\sim}{x}$, and the sequence $\vec{d}$ defining $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$. (Since we consider an iteration of Souslin proper forcing, we may assume that for all α, $\Vdash_\alpha$"d_α codes a Souslin proper forcing.")

Let $\bar{N}$ be the Mostowski collapse of N, and let $\vec{c}$ be the image of $\vec{d}$ under this map. For any $z \in N$, let $\overline{z}$ be the image of z under the collapsing map, $\overline{z} = \{\overline{y} : y \in z \cap N\}$. Let $S = N \cap \varepsilon$, so for every $\alpha \in S$, $\overline{\alpha} = \alpha^S$ (see 2.3).

$\bar{N}$ is a transitive countable set, so it is in some V_λ, $\lambda < \kappa$. In V_λ, define a sequence $\vec{e}$ of length ε^S by requiring for all $\alpha \in S$:

$$\Vdash_{P_{\vec{e}\restriction\alpha^S}} \text{"If } G\restriction(\alpha^S, \bar{N}) \text{ is generic for } P^{\bar{N}}_{\vec{c}\restriction\alpha^S}\text{, then } e_{\alpha^S} = c_{\alpha^S}[G\restriction(\alpha^S, \bar{N})]\text{"}$$

(and $e_{\alpha^S} = \emptyset$ otherwise) (here G stands for G_P, where $P = P_{\vec{e}\restriction\alpha^S}$).

So we have three models: $\bar{N} \subset V_\lambda \subset V_\kappa$, and three definitions of iterations: $\vec{d}$ codes an iteration of length ε in V_κ (or in N), $\vec{e}$ codes an iteration of length ε^S in V_λ, and $\vec{c}$ codes an iteration of length ε^S in $\bar{N}$.

We will use Lemma 2.16 twice: in V_λ for $\vec{c}$ and $\vec{e}$, and in V_κ for $\vec{e}$ and $\vec{d}$.

First we use the lemma in V_λ to obtain a $(\overline{p}, 0, \bar{N}, \varepsilon^S, \vec{c}, \vec{e})$-generic condition $q' \in P^{V_\lambda}_{\vec{e}}$. Then we use the lemma in V_κ to obtain a $(q', 0, V_\lambda, S, \vec{e}, \vec{d})$-generic condition $q \in P^{V_\kappa}_{\vec{d}}$. Now we let $G_\varepsilon \subseteq P_{\vec{d}}$ be generic over V_κ, containing q. We let for $\alpha \leq \varepsilon$, $\alpha \in S$

$$\begin{aligned} G'_\alpha &= G_\alpha\restriction(S, V_\lambda, \vec{e}\restriction\alpha^S, \vec{d}\restriction\alpha) \quad \text{computed in } V_\kappa[G_\alpha], \\ G''_\alpha &= G'_\alpha\restriction(\alpha^S, \bar{N}, \vec{c}\restriction\alpha^S, \vec{e}\restriction\alpha^S) \quad \text{computed in } V_\lambda[G'_\alpha], \\ \overline{G}_\alpha &= \{\overline{p} : p \in G_\alpha \cap N\}. \end{aligned}$$

(Strictly speaking, G''_α is only well-defined if G'_α is generic for $P^{V_\lambda}_{\vec{e}\restriction\alpha^S}$.)

We claim that for all $\alpha \in (\varepsilon + 1) \cap S$, (1)–(11) hold.

(1) G'_α is generic for $P_{\vec{e}\restriction\alpha^S}$ over V_λ and contains $q'\restriction\alpha^S$. (Thus G''_α is well defined.)

(2) G''_α is generic for $P_{\vec{c}\restriction\alpha^S}$ over $\bar{N}$ and contains $\overline{p}\restriction\alpha^S$.

(3) If $\alpha = \beta + 1$, then $\overline{G}_\alpha = \overline{G}_\beta * \big(G(\beta) \cap \overline{N}[\overline{G}_\beta]\big)$.

(4) $\overline{G}_\alpha = G''_\alpha$.

(5) $G_\alpha \cap N$ is generic for $P^N_{\vec{d}\restriction\alpha}$ over N.

(6) For all $P_{\vec{d}\restriction\alpha}$-names $\underset{\sim}{y}$ of reals that are in N, $\underset{\sim}{y}[G_\alpha] = \underset{\sim}{\overline{y}}[\overline{G}_\alpha]$.

(7) $c_{\alpha^S}[G''_\alpha] = d_\alpha[G_\alpha]$.

(8) $V_\kappa[G_\alpha] \models$"$d_\alpha[G_\alpha]$ codes a Souslin proper forcing".

(9) $V_\lambda[G'_\alpha] \models$"$c_{\alpha^S}[G''_\alpha]$ codes a Souslin proper forcing".

(10) $c_{\alpha^S}[G''_\alpha] = e_{\alpha^S}[G'_\alpha]$.

(11) $V_\lambda[G'_\alpha] \models$"$e_{\alpha^S}[G_\alpha]$ codes a Souslin proper forcing."

The verification of this claim is a routine computation, (mainly a consequence of 2.10 and 2.16), but we will carry it out anyway. The proof is by induction on α.

Proof of (1): Let $\mu = \min\{\beta \leq \alpha : G'_\beta \text{ not generic}\} \cup \{\alpha+1\}$. By induction assumption (1), $\mu = \alpha$ or $\mu = \alpha + 1$.

If $\mu = \alpha$, then (since G_α contains a $(q', 0, V_\lambda, S, \vec{e}\restriction\alpha^S, \vec{d}\restriction\alpha)$-generic condition), by 2.10(A) we must have that α is a successor, and by 2.10(B) either $d_{\alpha-1}[G_{\alpha-1}] \neq e_{\alpha^S-1}[G'_{\alpha^S-1}]$ or $V_\kappa[G_\alpha] \models$"$d_{\alpha-1}[G_{\alpha-1}]$ does not code a Souslin proper forcing". Both possibilities contradict some inductive assumption ((7) or (8)).

Hence $\mu = \alpha + 1$, and G'_α is generic. By 2.10(C), G'_α contains $q'\restriction\alpha^S$.

Proof of (2): Almost verbatim the same as of (1), replacing G' by G'', V_λ by $\bar{N}$, etc. Instead of the inductive assumptions (7) and (8) we have to use (10) and (11).

Proof of (3): Every element of $\overline{G}_{\beta+1}$ is of the form $\bar{r}$ for some $r \in N$. We have

$$(*) \qquad \bar{r} \in \overline{G}_{\beta+1} \Leftrightarrow r \in G_{\beta+1} \Leftrightarrow r\restriction\beta \in G_\beta \,\&\, r(\beta)[G_\beta] \in G(\beta).$$

By (6), $r(\beta)[G_\beta] = \overline{r(\beta)}[\overline{G}_\beta]$. Since the map $x \to \bar{x}$ is an isomorphism,

$$\overline{r(\beta)} = \bar{r}(\bar{\beta}) = \bar{r}(\beta^S) \quad \text{and} \quad \overline{r\restriction\beta} = \bar{r}\restriction\beta^S.$$

Therefore the chain of equivalences in $(*)$ can be continued by

$$\ldots \Leftrightarrow \bar{r}\restriction\beta^S \in \overline{G}_\beta \,\&\, \bar{r}(\beta^S)[\overline{G}_\beta] \in G(\beta) \Leftrightarrow \bar{r} \in \overline{G}_\beta * \left(G(\beta) \cap \bar{N}[\overline{G}_\beta]\right).$$

Proof of (4): Note that both $\overline{G}_\alpha$ and G''_α are subsets of $\bar{N}$. First assume that α is a limit.

Then for $r \in N$:

$$\bar{r} \in \overline{G}_\alpha \Leftrightarrow r \in G_\alpha \Leftrightarrow \forall\beta \in S \cap \alpha \; r\restriction\beta \in G_\beta.$$

(The last $\Leftarrow$ holds because $S \cap \alpha = N \cap \alpha$ is cofinal in the support of r.)

$$\ldots \Leftrightarrow \forall\beta \in S \cap \alpha \; \overline{r\restriction\beta} = \bar{r}\restriction\beta^S \in \overline{G}_\beta = G''_\beta \Leftrightarrow \bar{r} \in G''_\alpha.$$

(The last equivalence holds by 2.6 and induction hypotheses (1), (8) and (11).)

For $\alpha = \beta + 1$, by Definition 2.6 we have

$$G''(\beta^S) = G'(\beta^S) \cap \bar{N}[G''_\beta],$$
$$G'(\beta^S) = G(\beta) \cap V_\lambda[G'_\beta],$$

hence

$$G''_{\beta+1} = G''_\beta * \left(G(\beta) \cap \bar{N}[G''_\beta]\right) = \overline{G}_\beta * \left(G(\beta) \cap \bar{N}[\overline{G}_\beta]\right)$$

by induction hypothesis, so by (3) the proof is finished.

Proof of (5): Let $D \in N$ be a dense set in $P^N_{\vec{d}\restriction\alpha}$, then $\overline{D}$ is dense in $P^{\bar{N}}_{\vec{c}\restriction\alpha^S}$. By (2) and (4), there exists a condition $\bar{r} \in \bar{N} \cap \overline{D} \cap \overline{G}_\alpha$. Hence $r \in N \cap D \cap G_\alpha$.

Proof of (6): Assume $\underset{\sim}{y}[G_\alpha](n) = i$. Then (by (5)), there exists a condition $r \in G_\alpha \cap N$ such that

$$V_\kappa \text{ or } N \models r \Vdash \underset{\sim}{y}(\check{n}) = \check{\imath}.$$

So

$$\bar{N} \models \bar{r} \Vdash \underset{\sim}{\bar{y}}(\check{n}) = \check{\imath}.$$

(Note that $\bar{n} = n$.) Since $\bar{r} \in \overline{G}_\alpha$, $\underset{\sim}{\bar{y}}[\overline{G}_\alpha](n) = i$.

(7) is just a special case of (6).

(8) is true by definition of the sequence $\vec{d}$. (No inductive proof needed for this fact.)

(9) follows from (7), (8) and 1.7.

(10) follows from (9), by the definition of $\vec{e}$.

(11) follows from (9) and (10).

This completes the proof of (1)–(11).

To finish the proof of 3.1, note that (5) implies that $\underset{\sim}{x}[G_\alpha] = \underset{\sim}{\bar{x}}[G''_\alpha]$ can be computed in $V_\lambda[G'_\alpha]$, and since $\bar{p} \in \overline{G}_\epsilon$, $p \leq^* q$. ∎

Note that we did not prove that $\vec{e}$ and $\vec{c}$ are corresponding sequences. Although (9) and (10) do not explicitly mention $\vec{c}$, G_α and $\vec{d}$, the proof uses the fact that G'_α was obtained as the restriction of a filter G_α on $P_{\vec{d}\restriction\alpha}$.

4. Conclusion: Projective Measurability and the Borel Conjecture

4.1 Notation: Using the notation from 3.1, let Q_{κ^+} be the collapse of κ^+ ($\doteq\aleph_2$) to κ ($=\aleph_1$) in $V_{\kappa^+} := V[H_\kappa * G_{\kappa^+}]$ (so the conditions are countable$^{V_{\kappa^+}}$ partial functions from κ to κ^+).

Let $G(\kappa^+) \subseteq Q_{\kappa^+}$ be generic over V_{κ^+}, $V_{\kappa^++1} := V_{\kappa^+}[G(\kappa^+)]$.

4.2 Fact: $\mathbb{R} \cap V_{\kappa^+} = \mathbb{R} \cap V_{\kappa^++1}$.

Proof: Q_{κ^+} is σ-closed. ∎

4.3 LEMMA: *Let x be a real in V_{κ^+}, and let $H_0' \subseteq R_{\lambda_0}$ be generic, $\lambda_0 < \kappa$, $H_0' \in V_{\kappa^+}$.*

Then there exists a generic $H_\kappa' \subseteq R_\kappa$ in V_{κ^+} such that $x \in V[H_\kappa']$ and $H_0' \subseteq H_\kappa'$. (Clearly then, for some $\lambda_1 < \kappa$, letting $H_1' = H' \restriction R_{\lambda_1}$, we have $x \in V[H_1']$.)

Proof: Since H_0' is countable, we can use 3.1 to get a generic $G_\kappa' \subseteq R_\kappa$ such that x and H_0' are in $V[G_\kappa']$. $V[G_\kappa']$ is a forcing extension of $V[H_0']$ via $\mathrm{Coll}(\kappa)^{V[H_0']} \approx \mathrm{Coll}(\kappa)^{V_0}/H_0'$. So $V[G_\kappa'] = V[H_0' * H'']$ for some H''. Let $H_\kappa' := H_0' * H''$. ∎

4.4 Fact: $H_\kappa \subseteq R_\kappa$ is generic (over V) iff for all $\lambda < \kappa$, $H_\kappa \cap R_\lambda$ is generic.

Proof: Any maximal antichain of R_κ has size $< \kappa$, so it is contained in some R_λ and hence meets H_λ. ∎

4.5 THEOREM: *Assume H_κ, G_{κ^+}, $G(\kappa^+)$ are as in 4.1. Then*

$$(*) \quad \exists H_\kappa' \in V_{\kappa^++1}, H_\kappa' \subseteq R_\kappa \ (H_\kappa' \ V_0\text{-generic}),\ \mathbb{R} \cap V[H_\kappa'] = \mathbb{R} \cap V[G_{\kappa^+}].$$

Proof: V_{κ^++1} and V_{κ^+} have the same reals. In V_{κ^++1}, enumerate all the reals $\langle r_\xi : \xi < \kappa \rangle$ and find (using 4.3) an increasing sequence of ordinals $\langle \lambda_\xi : \xi < \kappa \rangle$ and an increasing sequence $\langle H_\xi : \xi < \kappa \rangle$, where all H_ξ are in V_{κ^+}, $H_\xi \subseteq R_{\lambda_\xi}$ generic, $r_\xi \in V[H_\xi]$.

Then $H_\kappa = \bigcup H_\xi \subseteq R_\kappa$ is generic, and $V_0[H_\kappa]$ contains all the reals of $V[G_{\kappa^+}]$. Conversely, every real in $V[H]$ is in some $V[H_\xi]$ and hence in $V[G_{\kappa^+}]$.

4.6 COROLLARY: *For any closed first order statement φ about the reals (without parameters),*

$$V_\kappa \models \varphi \ \Leftrightarrow \ V_{\kappa^+} \models \varphi.$$

Proof: Assume $V_\kappa = V[H_\kappa] \models \varphi$. Then, (by [15]) $0 \Vdash_{R_\kappa} \varphi$ (where 0 is the weakest condition in R_κ). This implies $V[H_\kappa'] \models \varphi$, and therefore $V[G_{\kappa^+}] \models \varphi$.

4.7 Remark: As the statement "All Σ_n^1-sets are measurable" is itself a first order statement, the above argument proves projective measurability in V_{κ^+}.

4.8 APPLICATION 1: *If* Con(ZFC+IC), *then* Con(ZFC+PM+BC).

Proof: Assume that κ is inaccessible. We start from $V = L$, collapse κ, and iterate κ^+ many Mathias (or Laver) reals. By [1] or [10], V_{κ^+} satisfies BC, and by 4.7, V_{κ^+} satisfies PM. ∎

4.9 COROLLARY:

(1) Con(ZFC+PM+BC) *iff* Con(ZFC+PM+¬BC) *iff* Con(ZFC+IC).

(2) Con(ZFC+¬PM+BC) *iff* Con(ZFC+¬PM+¬BC) *iff* Con(ZFC).

Proof of the Corollary: (1) Note that Solovay's model satisfies CH if the ground model is L, and CH implies ¬BC.

Hence Con(ZFC+IC) implies Con(ZFC+PM+¬BC).

The only other implication that does not follow from Solovay's and Shelah's theorems ([15] and [14]) is the one from 4.8.

(2) L will always satisfy ¬PM + ¬BC. Starting with a model of "$V = L$ and there is no inaccessible cardinal", Laver's construction will yield a generic extension satisfying BC+¬PM (by [14]). ∎

4.10 APPLICATION 2: *If*

Con(ZFC+IC), *then* Con(ZFC+PM+MA(Souslin proper)+$\mathfrak{c} = \aleph_2$).

Proof: Similar to application 1, using 1.18. ∎

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